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Second-harmonic resonance in the interaction of an air stream with capillary–gravity waves

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The method of multiple scales is used to derive equations governing the temporal and spatial variation of the amplitudes and phases of inviscid capillary–gravity travelling waves in the case of second-harmonic resonance (Wilton's ripples), but including the effects of: (i) near resonance, (ii) liquid depth, and (iii) pressure perturbations exerted by an external subsonic gas on the liquid/gas interface. The spatial form of the equations shows that, below a critical gas velocity, energy is transferred between the fundamental and its first harmonic in keeping with the energy conservation law. However, the amplitude of the first harmonic decreases with increasing gas velocity. Above the critical gas velocity, the displacement of the gas/liquid interface grows monotonically with distance. It is found that pure amplitude-modulated waves are possible only at perfect resonance. Pure phase-modulated, near-resonant waves are periodic, as the resonance forces a readjustment of the phases to produce perfect resonance. The effectiveness of the resonance in rippling the interface increases as the liquid depth decreases.

1. Introduction

In this paper, we consider the second-harmonic resonance in the interaction of capillary and gravity waves on the interface of an inviscid liquid layer of finite depth and a gas flowing uniformly parallel to the interface. The condition of nth-harmonic resonance occurs when both the fundamental and its nth harmonic propagate in the same direction with identical phase speeds.

In the absence of an external gas and for a deep liquid, harmonic resonance occurs at the denumerable set of wavenumbers $\tilde{k}_n = (\rho g/nT)^{\frac{1}{2}}$, $n \geqslant 2$, where T is the surface tension, ρ is the density of the liquid and g is the acceleration due to gravity. The second-harmonic resonance wavenumber $\tilde{k}_2$ corresponds to a wavelength of 2·44 cm in deep water (Wilton 1915), while the third-harmonic resonance wavenumber $\tilde{k}_3$ corresponds to a wavelength of 2·99 cm in deep water. At $\tilde{k}_2$, Wilton found that two finite amplitude permanent periodic waves could exist; one is a gravity-like wave in which the phase speed decreases with amplitude, while the second is a capillary-like wave in which the phase speed decreases with amplitude. Pierson & Fife (1961) extended the results of Wilton to wavenumbers near $\tilde{k}_2$, while Schooley (1960) observed double-dimpled wave profiles. Barakat & Houston (1968) extended the analysis of Pierson & Fife to the case of a finite depth liquid while Nayfeh (1970*a*) extended the results of Barkat & Houston to second order. At or near $\tilde{k}_3$, Nayfeh (1970*b*) found that, to third order, three finite amplitude periodic waves could exist: one is gravity-like while the others are capillary-like.

At $\tilde{k}_2$, Simmons (1969) derived equations governing the temporal and spatial variation of the amplitudes and phases of the fundamental and its first harmonic by averaging the Lagrangian for the case of a deep liquid. From a one-dimensional form of these equations, Simmons found that pure amplitude-modulated waves could exist. These waves were confirmed experimentally by McGoldrick (1970*a*). McGoldrick (1970*b*) rederived the equations of Simmons, using the method of multiple scales, and found that the general motion consists of both amplitude- and phase-modulated waves. Moreover, he found that the periodic waves of Wilton and the pure amplitude-modulated waves of Simmons exist for very special initial conditions only.

At or near $\tilde{k}_3$, Nayfeh (1971) derived equations governing the temporal and spatial variation of the amplitudes and phases of the fundamental and its second harmonic for the case of deep water. The results show that pure amplitude-modulated waves do not exist in this case, while pure phase-modulated waves are unstable.

Kim & Hanratty (1971) and McGoldrick (1972) presented experimental results on harmonic resonances. Kim & Hanratty observed the creation of third-, fourth- and eighth-harmonic distortions in shallow water. They interpreted these observations by using a quadratic interaction model consisting of four modes. McGoldrick determined experimentally the effect of near resonance for third-, fourth- and sixth-harmonic resonance in deep water.

The purpose of this paper is to analyse the effect of a subsonic air stream on rippling of the surface of an adjacent liquid of finite depth near the second-harmonic resonance conditions.

2. Problem formulation

In this paper we consider the flow configuration analysed by Nayfeh & Saric (1971). The liquid is assumed to be inviscid, and to have a finite depth, but to be otherwise unlimited. One face of the liquid is assumed to be adjacent to an inviscid subsonic gas of density ρ_g flowing with a uniform velocity U_g parallel to the undisturbed liquid/gas interface. The density of the gas is assumed to be very small compared with the liquid density so that the gas body force can be neglected. Moreover, the phase velocity is assumed to be small (of the order of 20 cm/s) compared with the gas velocity (of the order of m/s) so that the transient motion of the gas can be neglected. The motion is limited to two dimensions, and it is assumed to be represented by potential functions.

A Cartesian co-ordinate system is introduced such that the x axis lies in the plane of the undisturbed liquid/gas interface, and the y axis normal to this interface and directed from the liquid to the gas. Distances and time are made dimensionless using the wavenumber $k_c = (\rho g/T)^{\frac{1}{2}}$ and the time $(gk_c)^{-\frac{1}{2}}$, where g is the acceleration due to gravity and T is the surface tension of the liquid. The dimensional potential functions representing the oscillations of the liquid and the gas are taken to be

$$g^{\frac{1}{2}}k_c^{-\frac{3}{2}}\phi(x,y,t), \quad U_g[x+\Phi(x,y,t)]/k_c,$$

where the dimensionless functions ϕ and Φ satisfy

$$\nabla^2\phi = 0, \quad -h \leqslant y < \eta, \tag{1}$$

and

$$\Phi_{yy}+m^2\Phi_{xx} = M^2[\tfrac{1}{2}(\gamma-1)(2\Phi_x+\Phi_x^2+\Phi_y^2)(\Phi_{xx}+\Phi_{yy})+(2\Phi_x+\Phi_x^2)\Phi_{xx} + 2(1+\Phi_x)\Phi_y\Phi_{xy}+\Phi_y^2\Phi_{yy}] \quad (\eta \leqslant y < \infty), \tag{2}$$

for $-\infty < x < \infty$, where h is the depth of the liquid layer, $\eta(x,t)$ is the elevation of the wave above the undisturbed interface, M is the gas Mach number, γ is the specific heat ratio of the gas and $m^2 = 1-M^2$.

At the solid/liquid interface, the normal velocity vanishes, that is,

$$\phi_y(x,-h,t) = 0, \tag{3}$$

and away from the gas/liquid interface, the vertical component of the gas velocity vanishes, that is,

$$\Phi_y(x,\infty,t) = 0. \tag{4}$$

At the liquid/gas interface, the normal components of the gas and the liquid velocities are equal to each other and to the normal velocity of the interface itself; that is

$$\left.\begin{aligned} \eta_t+\phi_y &= \eta_x\phi_x \quad (5)\\ \eta_x-\Phi_y &= -\eta_x\Phi_x \quad (6)\end{aligned}\right\} \quad \text{at} \quad y=\eta.$$

Moreover, the balance of normal forces on this interface gives

$$\eta-\phi_t+\tfrac{1}{2}(\phi_x^2+\phi_y^2) = \eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}}-\tfrac{1}{2}m\chi C_p \quad \text{at} \quad y=\eta, \tag{7}$$

where $\frac{1}{2}m\chi C_p$ is the dimensionless pressure perturbation exerted by the gas on the interface owing to the appearance of waves on the interface. Here,

$$\chi = \rho_g U_g^2 k_c/m\rho g$$

is the ratio of the gas pressure perturbation to the body force, and

$$C_p = (2/\gamma M^2)\{[1-\tfrac{1}{2}(\gamma-1)M^2(2\Phi_x+\Phi_x^2+\Phi_y^2)]^{\gamma/(\gamma-1)}-1\}$$

the pressure perturbation coefficient. In this model, the gas motion is energetically coupled with the liquid motion via the term $\frac{1}{2}m\chi C_p$, and the coupling disappears when $\chi \to 0$.

To determine an approximate solution to (1)–(7) for small ϵ (the maximum slope of the wave), we use the method of multiple scales (Nayfeh 1973, chapter 6) and introduce the temporal scales

$$T_0 = t, \quad T_1 = \epsilon t$$

and the spatial scales $\quad X_0 = x, \quad X_1 = \epsilon x.$

Thus, the derivatives are transformed according to

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0}+\epsilon\frac{\partial}{\partial T_1}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial X_0}+\epsilon\frac{\partial}{\partial X_1}.$$

Moreoever, we assume that

$$\eta(x,t;\epsilon) = \sum_{n=1}^{2} \epsilon^n \eta_n(X_0, X_1, T_0, T_1) + O(\epsilon^3), \tag{8}$$

$$\phi(x,y,t;\epsilon) = \sum_{n=1}^{2} \epsilon^n \phi_n(X_0, X_1, y, T_0, T_1) + O(\epsilon^3), \tag{9}$$

$$\Phi(x,y,t;\epsilon) = \sum_{n=1}^{2} \epsilon^n \Phi_n(X_0, X_1, y, T_0, T_1) + O(\epsilon^3). \tag{10}$$

Since the boundary conditions (3) and (4) are linear, each ϕ_n satisfies (3), while each Φ_n satisfies (4).

On substituting the expansions (8)–(10) into (1)–(7) and equating the coefficients of like powers of ϵ, we get to order ϵ

$$\frac{\partial^2\phi_1}{\partial X_0^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0, \quad m^2\frac{\partial^2\Phi_1}{\partial X_0^2} + \frac{\partial^2\Phi_1}{\partial y^2} = 0, \tag{11a, b}$$

$$\frac{\partial\eta_1}{\partial T_0} + \frac{\partial\phi_1}{\partial y} = 0, \quad \frac{\partial\eta_1}{\partial X_0} - \frac{\partial\Phi_1}{\partial y} = 0 \quad \text{at} \quad y = 0, \tag{11c, d}$$

$$\eta_1 - \frac{\partial\phi_1}{\partial T_0} - \frac{\partial^2\eta_1}{\partial X_0^2} - m\chi\frac{\partial\Phi_1}{\partial X_0} = 0 \quad \text{at} \quad y = 0, \tag{11e}$$

and to order ϵ^2

$$\frac{\partial^2\phi_2}{\partial X_0^2} + \frac{\partial^2\phi_2}{\partial y^2} = -2\frac{\partial^2\phi_1}{\partial X_0\partial X_1}, \tag{12}$$

$$m^2\frac{\partial^2\Phi_2}{\partial X_0^2} + \frac{\partial^2\Phi_2}{\partial y^2} = -2m^2\frac{\partial^2\Phi_1}{\partial X_0\partial X_1} + M^2\left[(\gamma+1)\frac{\partial\Phi_1}{\partial X_0}\frac{\partial^2\Phi_1}{\partial X_0^2} + (\gamma-1)\frac{\partial\Phi_1}{\partial X_0}\frac{\partial^2\Phi_1}{\partial y^2} + 2\frac{\partial\Phi_1}{\partial y}\frac{\partial^2\Phi_1}{\partial X_0\partial y}\right], \tag{13}$$

$$\frac{\partial\eta_2}{\partial T_0} + \frac{\partial\phi_2}{\partial y} = \frac{\partial\phi_1}{\partial X_0}\frac{\partial\eta_1}{\partial X_0} - \eta_1\frac{\partial^2\phi_1}{\partial y^2} - \frac{\partial\eta_1}{\partial T_1} \quad \text{at} \quad y = 0, \tag{14}$$

$$\frac{\partial\eta_2}{\partial X_0} - \frac{\partial\Phi_2}{\partial y} = -\frac{\partial\eta_1}{\partial X_0}\frac{\partial\Phi_1}{\partial X_0} + \eta_1\frac{\partial^2\Phi_1}{\partial y^2} - \frac{\partial\eta_1}{\partial X_1} \quad \text{at} \quad y = 0, \tag{15}$$

$$\eta_2 - \frac{\partial\phi_2}{\partial T_0} - \frac{\partial^2\eta_2}{\partial X_0^2} - m\chi\frac{\partial\Phi_2}{\partial X_0} = -\frac{1}{2}\left(\frac{\partial\phi_1}{\partial X_0}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi_1}{\partial y}\right)^2 + \eta_1\frac{\partial^2\phi_1}{\partial y\partial T_0} + m\chi\eta_1\frac{\partial^2\Phi_1}{\partial X_0\partial y} + \tfrac{1}{2}m^3\chi\left(\frac{\partial\Phi_1}{\partial X_0}\right)^2 + \tfrac{1}{2}m\chi\left(\frac{\partial\Phi_1}{\partial y}\right)^2 + \frac{\partial\phi_1}{\partial T_1} + 2\frac{\partial^2\eta_1}{\partial X_0\partial X_1} + m\chi\frac{\partial\Phi_1}{\partial X_1} \quad \text{at} \quad y = 0. \tag{16}$$

3. Expansions

The sinusoidal, travelling wave solution of the first-order problem can be written as

$$\eta_1 = A(X_1, T_1)\exp(i\theta) + \bar{A}(X_1, T_1)\exp(-i\theta), \tag{17a}$$

$$\phi_1 = i\omega[A\exp(i\theta) - \bar{A}\exp(-i\theta)]\cosh[k(y+h)]/k\sinh kh, \tag{17b}$$

$$\Phi_1 = -(i/m)[A\exp(i\theta) - \bar{A}\exp(-i\theta)]\exp(-mky), \tag{17c}$$

where $\bar{A}$ is the complex conjugate of A, $\theta = kX_0 - \omega T_0$ and ω satisfies the dispersion relationship

$$\omega^2 = k(k^2 - k\chi + 1)\tanh kh.$$

Thus, the interface of the two fluids is stable or unstable according to whether ω is real or complex. If $\chi \leqslant 2$, ω is real for all values of k and the interface is stable. On the other hand, if $\chi > 2$, ω is complex for $k_{c1} < k < k_{c2}$ and real for $k \geqslant k_{c2}$ and $k \leqslant k_{c1}$, where

$$k_{c1}, k_{c2} = 0{\cdot}5\chi \pm (0{\cdot}25\chi^2 - 1)^{\frac{1}{2}},$$

the so-called cut-off wavenumbers separating stability from instability. These are the results of the linear Kelvin–Helmholtz instability problem (Chang & Russell 1965). In this paper, we assume that ω^2 is positive definite (i.e. $\chi < 2$, or $k_{c2} < k < k_{c1}$ if $\chi \geqslant 2$) so that equations (17) represent a uniform travelling wave train.

Harmonic resonance will occur for all wavenumbers k such that both (k, ω) and $(nk, n\omega)$ for some integer $n \geqslant 2$ satisfy the above dispersion relationship. The first resonant wavenumber k_1 corresponds to $n = 2$ and is the solution of

$$(4k_1^2 - 2k_1\chi + 1)\tanh 2k_1 h = 2(k_1^2 - k_1\chi + 1)\tanh k_1 h. \tag{18}$$

The resonant wavenumber k_1 is a function of both the liquid depth h and the ratio χ of the gas pressure perturbation to the gravitational force. As $h \rightarrow \infty$, $k_1^2 \rightarrow \frac{1}{2}$ for all χ. If $\chi = 3 \times 2^{-\frac{1}{2}}$, $k_1^2 = \frac{1}{2}$ for all h. For small h, $k_1 = \frac{1}{3}\chi$. Note that, as $h \rightarrow 0$, the waves are weakly dispersive; that is, all waves have approximately the same phase speed. The variation of k_1 with χ and h was calculated by Nayfeh & Saric (1971) and is shown in figure 1. In order that the configuration be stable according to the linear theory, both of the bracketed expressions in (18) must be positive. This condition restricts the values of χ to those less than $\chi_c = 3 \times 2^{-\frac{1}{2}}$ because $k_{c1} < k_1$ and $2k_1 < k_{c2}$ (i.e. an unstable configuration) if $\chi > \chi_c$, and $k_1 = k_{c1} = 0{\cdot}5k_{c2}$ (i.e. $\omega = 0$) if $\chi = \chi_c$. To determine the resonant interaction at or near k_1, we let

$$\eta_1 = \sum_{n=1}^{2} A_n(X_1, T_1)\exp(i\theta_n) + \text{c.c.}, \tag{19}$$

$$\phi_1 = i\sum_{n=1}^{2} \omega_n A_n \frac{\cosh k_n(y+h)}{k_n \sinh k_n h}\exp(i\theta_n) + \text{c.c.}, \tag{20}$$

$$\Phi_1 = -\frac{i}{m}\sum_{n=1}^{2} A_n \exp(i\theta_n - mk_n y) + \text{c.c.}, \tag{21}$$

where c.c. stands for the complex conjugate and

$$\theta_n = k_n X_0 - \omega_n T_0, \quad \omega_n^2 = k_n(k_n^2 - k_n\chi + 1)/C_n,$$
$$C_n = \coth k_n h, \quad k_2 \approx 2k_1, \quad \omega_2 \approx 2\omega_1.$$

Substituting for η_1, ϕ_1 and Φ_1 from (19)–(21) into (12)–(16), we get

$$\frac{\partial^2\phi_2}{\partial X_0^2} + \frac{\partial^2\phi_2}{\partial y^2} = 2\sum_{n=1}^{2} \omega_n \frac{\partial A_n}{\partial X_1}\frac{\cosh k_n(y+h)}{\sinh k_n h}\exp(i\theta_n) + \text{c.c.}, \tag{22}$$

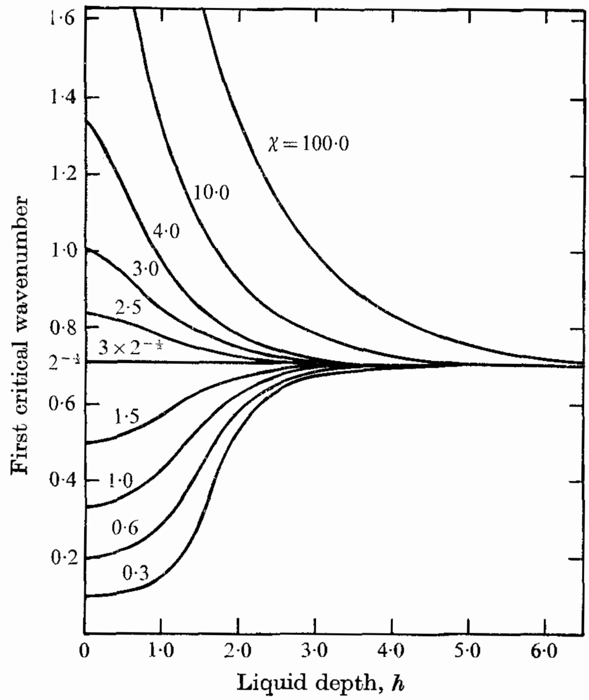

FIGURE 1. Variation of second-harmonic resonant wavenumber $k_1 = \tilde{k}_1/k_c$ with liquid depth $h = \tilde{h}k_c$ and $\chi = \rho_g U_g^2 k_c/\rho g(1-M^2)^{\frac{1}{2}}$, the ratio of pressure perturbation exerted by the gas to body force. $k_c = (\rho g/T)^{\frac{1}{2}}$, ρ is the density, T is the surface tension, g is the acceleration due to gravity, M is the Mach number, the subscript g refers to the gas and a tilde denotes a dimensional quantity.

$$m^2\frac{\partial^2\Phi_2}{\partial X_0^2}+\frac{\partial^2\Phi_2}{\partial y^2} = -2m\sum_{n=1}^{2} k_n\frac{\partial A_n}{\partial X_1}\exp(i\theta_n - mk_n y) + i(\gamma+1)\,k_1^3 M^2 m^{-2} A_1^2 \exp(2i\theta_1 - 2mk_1 y) + ik_1k_2(k_2-k_1)\,M^2 \times[(\gamma+1)\,m^{-2}-\gamma+3]\,A_2\bar{A}_1\exp[i(\theta_2-\theta_1)-m(k_1+k_2)\,y] + \text{c.c.} + \text{NST}, \quad (23)$$

$$\frac{\partial\eta_2}{\partial T_0}+\frac{\partial\phi_2}{\partial y} = -\sum_{n=1}^{2}\frac{\partial A_n}{\partial T_1}\exp(i\theta_n) - 2i\omega_1 k_1 C_1 A_1^2\exp(2i\theta_1) - i(k_2-k_1) \times(\omega_1C_1+\omega_2C_2)\,A_2\bar{A}_1\exp i(\theta_2-\theta_1) + \text{c.c.} + \text{NST} \quad \text{at} \quad y=0, \quad (24)$$

$$\frac{\partial\eta_2}{\partial X_0}-\frac{\partial\Phi_2}{\partial y} = -\sum_{n=1}^{2}\frac{\partial A_n}{\partial x_1}\exp(i\theta_n) - i(m+m^{-1})\,k_1^2A_1^2\exp(2i\theta_1) - im(k_2^2-k_1^2) \times A_2\bar{A}_1\exp i(\theta_2-\theta_1) + \text{c.c.} + \text{NST} \quad \text{at} \quad y=0, \quad (25)$$

$$\eta_2-\frac{\partial\phi_2}{\partial T_0}-\frac{\partial^2\eta_2}{\partial X_0^2}-m\chi\frac{\partial\Phi_2}{\partial X_0} = i\sum_{n=1}^{2}\left[\frac{\omega_nC_n}{k_n}\frac{\partial A_n}{\partial T_1}+(2k_n-\chi)\frac{\partial A_n}{\partial X_1}\right]\exp(i\theta_n) + [\tfrac{1}{2}\omega_1^2(3-C_1^2)-mk_1^2\chi]\,A_1^2\exp(2i\theta_1) + [\omega_1^2+\omega_2^2-\omega_1\omega_2(1+C_1C_2) - m\chi(k_2-k_1)^2]\,A_2\bar{A}_1\exp i(\theta_2-\theta_1) + \text{c.c.} + \text{NST} \quad \text{at} \quad y=0, \quad (26)$$

where NST stands for terms which do not produce secular terms.

Since
$$2\theta_1 = \theta_2-\Gamma, \quad \theta_2-\theta_1 = \theta_1+\Gamma,$$
where
$$\Gamma = (k_2-2k_1)\,X_0-(\omega_2-2\omega_1)\,T_0,$$

the terms proportional to $\exp[\pm(2i\theta_1)]$ and $\exp[\pm i(\theta_2-\theta_1)]$ are to within Γ resonant forcing functions. If we suppose a near-resonant tuning when k_2-2k_1 and $\omega_2-2\omega_1$ are of $O(\epsilon)$, then Γ becomes slowly varying on the same scales as A_1 and A_2. To exhibit this slow variation, we express Γ as

$$\Gamma = \frac{k_2-2k_1}{\epsilon}X_1 - \frac{\omega_2-2\omega_1}{\epsilon}T_1.$$

Consequently, additional secularities arise from the $\exp(\pm 2i\theta_1)$ and

$$\exp(\pm i[\theta_2-\theta_1])$$

terms. However, they can be balanced by proper choices of the functions $A_i(X_1, T_1)$.

To determine the conditions which must be satisfied for there to be no secular terms, we take the secular-free particular solution of (22)–(26) to be

$$\eta_2 = \sum_{n=1}^{2} F_n(X_1, T_1)\exp(i\theta_n) + \text{c.c.}, \tag{27}$$

$$\phi_2 = \sum_{n=1}^{2}\left[\omega_n G_n(X_1,T_1)\frac{\cosh k_n(y+h)}{k_n\sinh k_n h} + \omega_n(y+h)\frac{\partial A_n}{\partial X_1}\frac{\sinh k_n(y+h)}{k_n\sinh k_n h}\right]\exp(i\theta_n) + \text{c.c.}, \tag{28}$$

$$\begin{aligned}\Phi_2 = \sum_{n=1}^{2}\left[H_n(X_1,T_1) + y\frac{\partial A_n}{\partial X_1}\right]\exp(i\theta_n - mk_n y)\\ -\tfrac{1}{4}i(\gamma+1)k_1^2M^4m^{-3}yA_1^2\exp(2i\theta_1-2mk_1y)\\ +\tfrac{1}{4}iM^2m^{-2}(k_2-k_1)[(\gamma+1)m^{-2}-\gamma+3]A_2\bar{A}_1\\ \exp[i(\theta_2-\theta_1)-m(k_1+k_2)y] + \text{c.c.}\end{aligned} \tag{29}$$

Equations (28) and (29) satisfy (22) and (23) and the boundary conditions (3) and (4). Since this particular solution is free of secular terms, it is attainable if certain conditions are satisfied. These are the conditions for the elimination of secular terms.

Substituting this particular solution into (24)–(26) and equating the coefficients of $\exp(i\theta_n)$ on both sides, we get two sets of three algebraic inhomogeneous equations for the determination of F_i, G_i and H_i for $i = 1$ and 2. The determinants of both sets are zero because $\omega_i^2 = k_i(k_i^2-k_i\chi+1)/C_i$. Consequently, these algebraic equations are solvable if and only if the inhomogeneous part of each set is orthogonal to the solution of the homogeneous adjoint equations. This condition is equivalent to letting, say, $F_1 = F_2 = 0$ and eliminating the G's and H's from the six algebraic equations. In either case, we get the following equations as conditions for the elimination of secular terms:

$$\partial A_1/\partial T_1 + \omega_1'\,\partial A_1/\partial X_1 = iJ_1\bar{A}_1A_2\exp(i\Gamma), \tag{30}$$

$$\partial A_2/\partial T_1 + \omega_2'\,\partial A_2/\partial X_1 = iJ_2A_1^2\exp(-i\Gamma) \tag{31}$$

where $\omega_i' = d\omega_i/dk_i$ are the group velocities, $J_i = JC_i^{-1}2^{1-i}$ and

$$J = k_1\omega_i^{-1}[\tfrac{1}{2}(3-C_1^2-4C_1C_2)\,\omega_1^2 + m^{-1}k_1^2\chi + \tfrac{1}{4}(\gamma+1)M^4m^{-3}k_1^2\chi]. \tag{32}$$

In (32), we replaced k_2 by $2k_1$ and ω_2 by $2\omega_1$ with an error of $O(\epsilon)$. The complex conjugate counterparts of (30) and (31) do not add anything new.

In the case of no external air flow (i.e. $\chi = 0$),

$$J = \tfrac{1}{2}k_1\omega_1(3 - C_1^2 - 4C_1C_2), \quad \omega_i^2 = k_i(k_i^2+1)/C_i.$$

If we also assume that $k_2 = 2k_1$ and $\partial A_n/\partial X_1 = 0$, equations (30) and (31) correspond to equations (8) of Kim & Hanratty (1971) with $A_3 = A_1 = 0$ except for a missing minus sign in the first of their equations (8). Note that their L_2 corresponds to the present Γ. Their equations (32) remove the limitation $\partial A_n/\partial X_1 = 0$. Note that, as McGoldrick (1972) remarked, the validity of equations (8) of Kim & Hanratty is limited to the case in which each L_n is small, which is equivalent to $k_n = nk_1$ and $\omega_n \approx n\omega_1$ for $n = 1, 2, 3$ and 4. Thus, this assumption limits the validity of their equations (8) to shallow water; but then, omission of the higher harmonics is not justifiable because all harmonics have approximately the same phase speed.

If we let $A_n = \tfrac{1}{2}a_n \exp(i\beta_n)$ with a_n and β_n real and slowly varying in (30) and (31) and separate real and imaginary parts, we get

$$\partial a_1/\partial T_1 + \omega_1' \,\partial a_1/\partial X_1 = -\tfrac{1}{2}J_1 a_1 a_2 \sin\alpha, \tag{33}$$

$$\partial a_2/\partial T_1 + \omega_2' \,\partial a_2/\partial X_1 = \tfrac{1}{2}J_2 a_1^2 \sin\alpha, \tag{34}$$

$$\partial \beta_1/\partial T_1 + \omega_1' \,\partial \beta_1/\partial X_1 = \tfrac{1}{2}J_1 a_2 \cos\alpha, \tag{35}$$

$$a_2(\partial \beta_2/\partial T_1 + \omega_2' \,\partial \beta_2/\partial X_1) = \tfrac{1}{2}J_2 a_1^2 \cos\alpha, \tag{36}$$

where

$$\alpha = \beta_2 - 2\beta_1 + \frac{k_2 - 2k_1}{\epsilon} X_1 - \frac{\omega_2 - 2\omega_1}{\epsilon} T_1. \tag{37}$$

If we let (i) $h \to \infty$ (i.e. infinite depth), (ii) $\omega_2 = 2\omega_1$ and $k_2 = 2k_1$ (i.e. perfect resonance) so that $\Gamma = 0$, and (iii) $\chi \equiv 0$ (i.e. no external gas), equations (33)–(37) reduce to those of Simmons (1969).

Note that for some values of χ, h and M, the interaction parameters J_1 and J_2 vanish. As $h \to \infty$, J_1 and J_2 vanish when

$$\chi = 6 \times 2^{\frac{1}{2}} m^3[(\gamma+1)M^4 + 4m^2(m+1)]^{-1}, \tag{38}$$

which becomes $\chi \approx 1{\cdot}06$ at $M = 0$. For such cases, (33)–(36) show that, to second order, there is no interaction between the fundamental and its first harmonic.

Since there is no general solution yet available for (33)–(37) subject to general initial conditions, we next investigate the spatial variation of the amplitudes and phases; that is, $\partial a_n/\partial T_1 = \partial \beta_n/\partial T_1 = 0$ and $\omega_2 = 2\omega_1$.

4. Spatial variation of the amplitudes and phases

In this case, (33) and (34) have the integral

$$a_1^2 + \nu a_2^2 = E, \quad \nu = 2C_2\omega_2'/C_1\omega_1', \tag{39}$$

where E is a constant proportional to the total energy density in the two modes. This is simply a statement of the conservation of energy. According to this equation, the motion is bounded for all distances if ν is positive. If ν is negative, no conclusion can be drawn from this equation on whether the motion is bounded or unbounded. We shall show below that the motion is unbounded if the detuning is sufficiently small.

For a deep liquid, $k_1^2 = 0{\cdot}5$ and $k_2^2 = 2{\cdot}0$, hence

$$\nu = 2[7-2^{\frac{3}{2}}\chi]/[5-2^{\frac{3}{2}}\chi].$$

Thus, ν is negative when $\quad \frac{5}{4}2^{\frac{1}{2}} < \chi < \frac{7}{4}2^{\frac{1}{2}}.$

The upper bound must be replaced by $3\times 2^{-\frac{1}{2}}$, the value below which χ was restricted in §3. For deep water and at sea level, ν is negative when the wind speed U_g satisfies the inequality

$$6{\cdot}3\,\mathrm{m/s} < U_g < 6{\cdot}9\,\mathrm{m/s}.$$

To determine a second integral, we combine (35)–(37) into

$$a_2\,d\alpha/dX_1 = \sigma a_2 + (\tfrac{1}{2}J_2\omega_2'^{-1}a_1^2 - J_1\omega_1'^{-1}a_2^2)\cos\alpha, \tag{40}$$

where the detuning $\sigma = (k_2-2k_1)/\epsilon$. From (34), (39) and (40), we get the integral

$$a_1^2a_2\cos\alpha + \sigma\omega_2' J_2^{-1}a_2^2 = L \quad \text{if} \quad J_2 \neq 0, \tag{41}$$

where L is another constant of integration. Using these integrals, we rewrite (33) as

$$\left(\frac{d\xi}{dX_1}\right)^2 = \frac{EJ_2^2}{\omega_2'^2}\left[\xi(1-\nu\xi)^2 - \frac{1}{E}\left(\frac{L}{E}-\frac{\sigma\omega_2'}{J_2}\xi\right)^2\right] = G(\xi), \quad \xi = a_2^2/E. \tag{42}$$

The stability of the interface depends on the number of real roots of the algebraic cubic equation $G(\xi) = 0$. If $G(\xi)$ has only one real root, the displacement of interface becomes unbounded with increasing distance. On the other hand, if $G(\xi)$ has three real roots, the interface is bounded and ξ oscillates periodically between the two positive roots between which $G(\xi)$ is positive. In this case, ξ can be expressed in terms of Jacobi elliptic functions.

For the particular initial conditions

$$a_1^2 = E, \quad a_2 = 0 \quad \text{at} \quad x = X_1 = 0 \tag{43}$$

equation (41) leads to $L = 0$. Hence, (42) can be rewritten as

$$(d\xi/dX_1)^2 = R\xi[(1-\nu\xi)^2 - 2\mu\xi], \tag{44}$$

where $\quad \mu = \sigma^2\omega_2'^2/2EJ_2^2, \quad R = EJ_2^2/\omega_2'^2.$

Since $\xi = 0$ at $X_1 = 0$, equation (44) shows that the interface is bounded if

$$(1-\nu\xi)^2 - 2\mu\xi = 0 \tag{45}$$

has positive real roots; this condition is equivalent to

$$(\mu+\nu)^2 > \nu^2, \tag{46}$$

which is satisfied for all μ if $\nu > 0$ and for $\mu > 2|\nu|$ if $\nu < 0$. Thus, the interface is unstable and its displacement grows as the waves travel if the air speed $U_g > 6{\cdot}3\,\mathrm{m/s}$ (i.e. $\nu < 0$) and the detuning $\sigma = (k_2-2k_1)/\epsilon$ is small enough so that

$$\sigma^2 < 4|\nu|\,EJ_2^2/\omega_2'^2.$$

In particular, the interface is unstable when $\nu < 0$ and $\sigma = 0$ (i.e. perfect resonance).

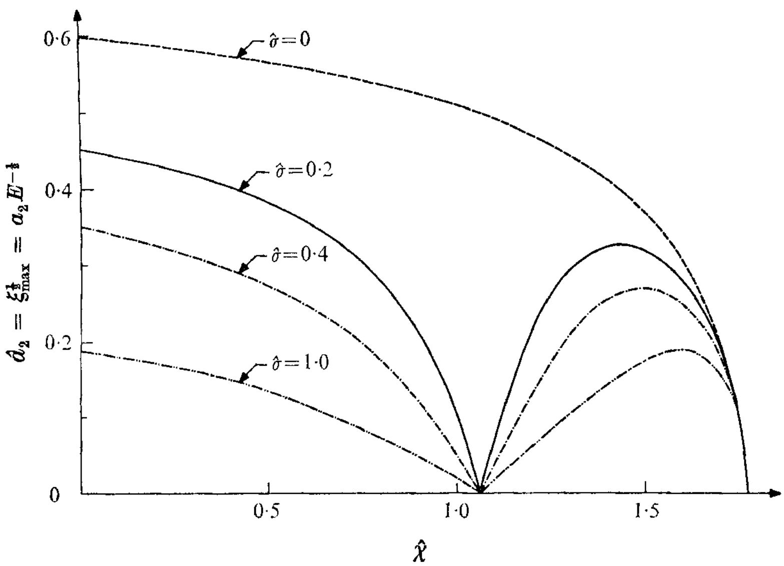

FIGURE 2. Effect of air flow on the effectiveness of resonance in rippling the interface for $M = 0$ and $h = \infty$.

If the quadratic equation (45) has real roots, ξ oscillates periodically between $\xi = 0$ (owing to the initial condition $\xi = 0$ at $x = 0$) and the smaller root of (45); that is, between $\xi = 0$ and

$$\xi = \xi_{\max} = \nu^{-2}[\mu+\nu-(\mu^2+2\mu\nu)^{\frac{1}{2}}]. \tag{47}$$

As $\mu \to \infty$ (i.e. $\sigma \to \infty$ or $J_2 \to 0$),

$$\xi_{\max} \to (\mu+\nu)^{-1}.$$

The rippling effect of the air on the liquid can be best visualized by plotting the maximum attainable dimensionless amplitude $\hat{a}_2 = a_2E^{-\frac{1}{2}}$ (i.e. $\xi^{\frac{1}{2}}_{\max}$) as a function of $\hat{\chi} = m\chi = \rho_g U_g^2 k_c/\rho g$ and the detuning parameter $\hat{\sigma} = \sigma E^{-\frac{1}{2}}$. Figure 2 shows that the presence of the air stream decreases the effectiveness of the resonance mechanism in rippling the interface. It shows that $\hat{a}_2$ decreases monotonically to zero as $\hat{\chi}$ increases to $\hat{\chi} \approx 1{\cdot}06$. As $\hat{\chi}$ increases further, $\hat{a}_2$ first increases and then decreases to zero when $\hat{\chi}$ satisfies $\mu = 2|\nu|$. Beyond this critical value, (45) has no real roots and the interface is unstable. Note that when $\sigma = 0$ (i.e. perfect resonance) $\hat{a}_2$ is discontinuous at $\hat{\chi} \approx 1{\cdot}06$, the value that makes $J_1 = J_2 = 0$.

Figure 3 shows that for a given $\hat{\chi}$ there exists a band of frequencies with bandwidth of $O(\epsilon)$ about the perfect resonance frequency for which $\hat{a}_2$ is appreciable, and the rippling is most effective. However, $\hat{a}_2$ decreases rapidly as $\hat{\sigma}$ increases as discussed above, in agreement with the conclusion and observations of McGoldrick (1972).

Figure 4 shows that the effectiveness of the resonance in rippling the interface increases as the liquid depth decreases, in agreement with the results of Kim & Hanratty (1971). For a given $\hat{\sigma}$, $\hat{a}_2$ increases as h decreases. Note that the present analysis is invalid for very small values of h.

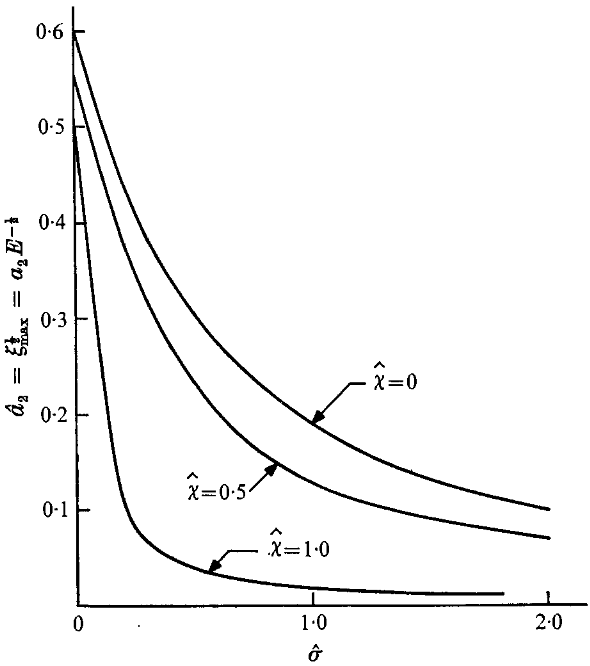

FIGURE 3. Effect of detuning on the effectiveness of resonance in rippling the interface for $M = 0$ and $h = \infty$.

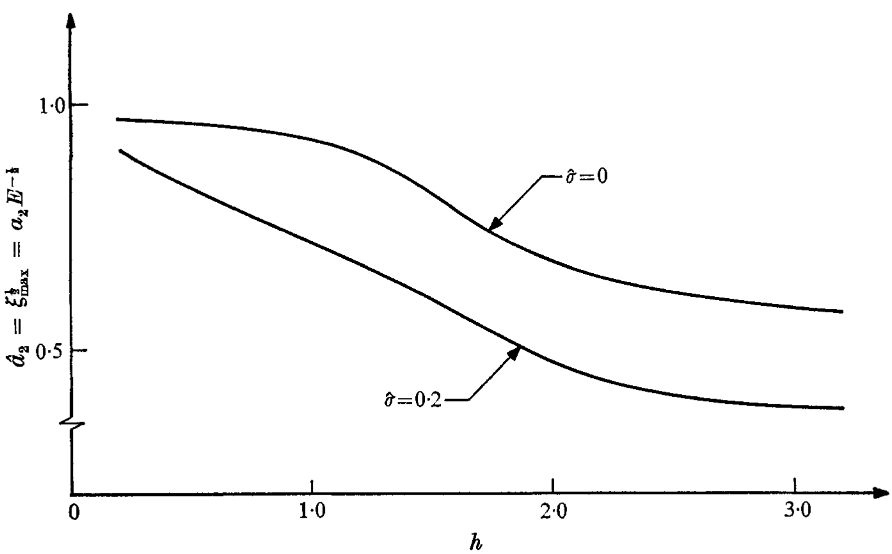

FIGURE 4. Effect of depth on the effectiveness of resonance in rippling the interface for $\hat{\chi} = 0·6$ and $M = 0$.

Periodic waves correspond to the stationary solutions of (33), (34) and (40); that is,

$$\left.\begin{aligned}\alpha_0 &= n\pi \text{ with } n \text{ an integer},\\ a_{20}+\tfrac{1}{2}J_2\omega_2'^{-1}a_{10}^2\cos n\pi - J_1\omega_1'^{-1}a_{20}^2\cos n\pi &= 0.\end{aligned}\right\} \tag{48}$$

Using (48), we rewrite (35) and (36) as

$$d\beta_1/dX_1 = -\tfrac{1}{2}J_1\omega_1'^{-1}a_{20}\cos n\pi,$$

$$d\beta_2/dX_1 = -\sigma + J_1\omega_1'^{-1}a_{20}\cos n\pi.$$

Consequently, $$k_2+\epsilon\, d\beta_2/dX_1 = 2(k_1+\epsilon\, d\beta_1/dX_1),$$

or $$\theta_2 = 2\theta_1.$$

Hence, the motion in this case is periodic, and the effect of the nonlinearity is to adjust the phase speed of the first harmonic to that of the fundamental. The dimensional phase speeds of periodic waves are given by

$$c = \left(\frac{Tg}{\rho}\right)^{\frac{1}{4}}\frac{\omega_1}{k_1}[1+\tfrac{1}{2}\epsilon(J_1/\omega_1'k_1)\,a_{20}\cos n\pi]+O(\epsilon^2). \tag{49}$$

Solving (39) and (48) for a_{10} and a_{20}, we get

$$\left.\begin{aligned}a_{20} &= \frac{\sigma\omega_1}{3J_1\cos n\pi}\left[1\pm\left(1+\frac{3EJ_1J_2}{\sigma^2\omega_1'\omega_2'}\right)^{\frac{1}{2}}\right],\\ a_{10} &= \pm(E-\nu a_{20}^2)^{\frac{1}{2}}.\end{aligned}\right\} \tag{50}$$

To determine the stability of these periodic waves, we let

$$a_1 = a_{10}+\Delta a_1\exp(sT_1),\quad a_2 = a_{20}+\Delta a_2\exp(sT_1),$$
$$\alpha = n\pi+\Delta\alpha\exp(sT_1).$$

Substituting these expressions into (33), (34) and (40), using (48), and keeping linear terms only, we find that s is pure imaginary. Hence, these periodic waves are unstable, in the sense that any small disturbance applied to these periodic waves leads to aperiodic waves.

So far, we have investigated two types of wave motion. The first consists of both amplitude- and phase-modulated waves while the second consists of pure phase-modulated waves (periodic waves). The question arises as to whether pure amplitude-modulated waves are possible. If β_1 and β_2 are constants, $\cos\alpha = 0$, or $\alpha = \frac{1}{2}(2n-1)$ with n integer according to (35) and (36). Hence, σ must vanish according to (37). Then, (33) and (34) have the solutions

$$\left.\begin{aligned}a_2 &= (E/\nu)^{\frac{1}{2}}\tanh[\pm\tfrac{1}{2}(EJ_1J_2/\omega_1'\omega_2')^{\frac{1}{2}}X_1+\text{constant}],\\ a_1 &= E^{\frac{1}{2}}\operatorname{sech}[\pm\tfrac{1}{2}(EJ_1J_2/\omega_1'\omega_2')^{\frac{1}{2}}X_1+\text{constant}].\end{aligned}\right\} \tag{51}$$

These equations reduce to those of Simmons (1969) when $h\to\infty$ and $\chi\to 0$. As $x\to\infty$, $X_1\to\infty$, $a_1\to 0$ and $a_2\to\pm(E/\nu)^{\frac{1}{2}}$, leading to a periodic wave independent of the fundamental. It can be shown that these waves are unstable because any small disturbance applied to such a motion would lead to a new motion consisting of both amplitude- and phase-modulated waves.

5. Concluding remarks

The method of multiple scales has been used to investigate the second-harmonic resonance (two-to-one resonances) in the interaction of capillary–gravity waves with a subsonic air stream. This case corresponds to a wave-length of 2·44 cm in deep water (Wilton's ripples). Equations that govern the temporal as well as the spatial variation of the amplitudes and phases of the two modes of oscillation are presented. Since there is no general solution yet available for these equations subject to arbitrary initial conditions, we investigated the spatial behaviour of the amplitudes and phases.

The spatial variation shows that $a_1^2 + \nu a_2^2 = E$ (in keeping with the principle of conservation of energy), where a_1 and a_2 are the amplitudes of the two modes, E is a constant and ν is a function of the liquid depth and air flow conditions. In the absence of the air stream and for an infinite depth, $\nu = 2{\cdot}8$, and hence the displacement of the liquid/gas interface is bounded for all distances. The same conclusions hold for any positive ν. The general motion in this case is an aperiodic travelling wave, and the air flow decreases the effectiveness of the second-harmonic resonance in rippling the interface. Pure amplitude-modulated waves are possible at exact resonance only, and to second order the energy is monotonically transferred from the fundamental to its first harmonic. Pure phase-modulated waves correspond to periodic waves near resonance, in which the nonlinear motion adjusts the phases to yield perfect resonance. The effectiveness of the resonance in rippling the interface decreases rapidly as the detuning increases, and increases as the liquid depth decreases. Note that the present theory is invalid for shallow water.

For certain air flow conditions (an air velocity greater than 6·3 m/s at sea level), ν is negative and the displacement of the interface may be unbounded depending on the relative magnitudes of the initial amplitudes and the detuning. At perfect resonance, the interface displacement becomes unbounded with increasing distance. This conclusion can be checked experimentally in a combination wave tank/wind tunnel.

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